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Inverse eigenvalue problem for a finite multiplying slab

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Abstract. This paper defines a general class of problem that has been termed the inverse eigenvalue problem. Basically similar problems have already been studied as isolated and specific examples in the analysis of time eigenvalues appearing in neutron transport theory. In this work, however, we present a general unified method for their treatment using functional analytic methods. Specifically, the critical slab problem has been analysed as an example of such an inverse eigenvalue problem of a Fredholm integral equation using the theory of perturbation of a class of positive, analytic operator-valued functions in Banach space. Numerical calculations of the critical thickness are given. These results are encouraging, considering the simplicity of the method, which does not involve an explicit solution of the Fredholm equation.

1. Introduction

Formal exact treatments of steady-state neutron transport in a multiplying and non-multiplying finite slab has been based on the singular eigenfunction technique of Case (Mitsis 1963, McCormick and Mendelson 1964), or on transform methods leading to singular integral equations which are adjoint to those obtained in the application of Case's method (Leonard and Mullikin 1964, Bowden *et al* 1968). In the original formulation of the problem based on the Case eigenfunction expansion, Mitsis (1963) considered the critical slab problem and derived a set of two coupled Fredholm integral equations for the discrete and continuum coefficients of Case's theory. These equations were then solved by iteration, that is, by expansion in Neumann series. In practice, due to the complexity of the equations, only a solution to zeroth and first order could be constructed. In neither of these approximations were the integral equations actually solved: the unknown function was at best approximated by the free inhomogeneous term of the equation. McCormick and Mendelson (1964) in studying the slab albedo problem followed an approach very similar to that of Mitsis, and constructed a slightly different iterative scheme for the solution of the coupled Fredholm equations in a Neumann series. Since the standard condition for the convergence of a Neumann series—the modulus of the eigenvalue in $\mathbb{K}\psi = \lambda\psi$ be greater than the spectral radius of the operator—is sufficient rather than necessary for the existence of a unique solution of the integral equation, the Neumann series criterion underestimates the range of validity of the Fredholm equations for the problem. Both Leonard and Mullikin (1964) and Bowden *et al* (1968) continue this approach through the construction of suitable Neumann series for their integral equations. In the present paper we start from the integral equations obtained by the application of Case's method (Mitsis 1963, McCormick and Mendelson 1964) but differ from the above works in the following

important respects: (i) we do not consider the two equations for the discrete and continuum coefficients separately, as has been done earlier, but combine them to form a single Fredholm equation, and (ii) analyse this equation functionally based on the theory of perturbation of linear operator in a Banach space (Kato 1966) and obtain good estimates of the perturbation parameter for a multiplying slab. The evaluation of the perturbation parameter defines an inverse eigenvalue problem in which one obtains the operator given an eigenvalue. We prove that the solution of the inverse eigenvalue problem in a steady state for a multiplying medium in slab geometry is unique. This gives a unique value for the perturbation parameter which, when the eigenvalue is unity, is the critical thickness of the slab. We therefore do not solve the integral equation specifically, and thus do not need the convergence criterion of the Neumann series.

2. The Fredholm equations

Consider a slab of thickness $2b$ with the origin of the coordinate system located at the centre. As in Leonard and Mullikin (1964), we consider both the critical and albedo problems, that is we study the one speed equation

$$\mu \frac{\partial \psi(x, \mu)}{\partial x} + \psi(x, \mu) = \frac{c}{2} \int_{-1}^1 \psi(x, \mu') d\mu' \quad (1)$$

subject to the slab albedo boundary conditions

$$\psi(-b, \mu) = \delta(\mu - \mu_0), \quad \mu, \mu_0 > 0 \quad (2a)$$

$$\psi(b, \mu) = 0, \quad \mu < 0 \quad (2b)$$

and note that the critical problem is obtained by putting $\delta(\mu - \mu_0) \equiv 0$. The solution of equation (1) in standard notation (Case and Zweifel 1967), is

$$\begin{aligned} \psi(x, \mu) = & a_{0+} \exp(-x/\nu_0) \phi_{0+}(\mu) + a_{0-} \exp(x/\nu_0) \phi_{0-}(\mu) \\ & + \int_{-1}^1 A(\nu) \exp(-x/\nu) \phi_{\nu}(\mu) d\nu. \end{aligned}$$

Use of the boundary conditions (2a) and (2b) and application of the half-range orthogonality relations (Case and Zweifel 1967) gives the two coupled Fredholm equations:

$$\begin{aligned} b_{\pm} = & [\nu_0 X(-\nu_0) (\exp(2z_0/\nu_0) \pm \exp(-2b/\nu_0))]^{-1} \\ & \times \left[\frac{2}{c} \nu(\mu_0) \pm \int_0^1 \nu X(-\nu) \exp(-2b/\nu) B_{\pm}(\nu) d\nu \right], \end{aligned} \quad (3)$$

$$\begin{aligned} B_{\pm}(\nu) = & f(c, \nu) \left(\frac{2}{c\nu} \gamma(\mu_0) \phi_{\nu}(\mu_0) - b_{\pm} X(\nu_0) \phi_{0+}(\nu) \mp b_{\pm} X(-\nu_0) \exp(-2b/\nu_0) \phi_{0-}(\nu) \right. \\ & \left. \mp \int_0^1 X(-\nu') \exp(-2b/\nu') \phi_{\nu'}(-\nu) B_{\pm}(\nu') d\nu' \right), \end{aligned} \quad (4)$$

where the new coefficients b_{\pm} and $B_{\pm}(\nu)$ are related to the original $a_{0\pm}$ and $A(\nu)$ by

$$b_{\pm} = (a_{0+} \pm a_{0-}) \exp(b/\nu_0), \quad (5)$$

$$B_{\pm}(\nu) = [A(\nu) \pm A(-\nu)] \exp(b/\nu). \quad (6)$$

$z_0(c)$ is the extrapolated endpoint defined as

$$z_0(c) = -\frac{1}{2}\nu_0 \ln[-X(-\nu_0)/X(\nu_0)]$$

and

$$\gamma(\mu) = \frac{c}{2(1-c)} \frac{\mu}{X(-\mu)(\nu_0^2 - \mu^2)},$$

$$f(c, \nu) = (1-c)g(c, \nu)X(-\nu)(\nu_0^2 - \nu^2)$$

where

$$g(c, \nu) = [(1 - c\nu \tanh^{-1} \nu)^2 + (\pi c\nu/2)^2]^{-1}, \quad (7)$$

and $X(\nu)$ is the well-known X -function of Case's theory and is defined by equation (25) below.

At this point, we depart from the standard (Mitsis 1963, McCormick and Mendelson 1964) and combine (3) and (4) to obtain the single non-homogeneous Fredholm equation for $B_{\pm}(\nu)$

$$B_{\pm}(\nu) = f(c, \nu) \left[\frac{2}{c\nu} \gamma(\mu_0) \phi_{\nu}(\mu_0) + \gamma(\mu_0) h_{\pm}(c, \nu, b) \right. \\ \left. \mp \frac{c}{2} \int_0^1 X(-\nu') \exp(-2b/\nu') \left(\frac{1}{\nu + \nu'} + h_{\pm}(c, \nu, b) \right) B_{\pm}(\nu') d\nu' \right] \quad (8)$$

in which

$$h_{\pm}(c, \nu, b) = \frac{\exp(2z_0/\nu_0)/(\nu_0 - \nu) \mp \exp(-2b/\nu_0)/(\nu_0 + \nu)}{\exp(2z_0/\nu_0) \pm \exp(-2b/\nu_0)}. \quad (9)$$

For the critical problem

- (i) there are no incident neutrons, i.e. $\gamma(\mu_0) = 0$
- (ii) $c > 1$, i.e. ν_0 is imaginary, $\nu_0 = ik_0$
- (iii) because of the symmetry of the problem

$$a_{0+} = a_{0-}, \quad A(\nu) = A(-\nu)$$

which implies $b_- = 0$, $B_-(\nu) = 0$. Call $B_+(\nu) \equiv B(\nu)$. Then

$$h_+(c, \nu, b) = -\frac{\nu + k_0 \tan((b + z_0)/k_0)}{\nu + k_0^2}$$

and equation (8) takes the form

$$B(\nu) = \frac{c}{2} \bar{f}(c, \nu) \int_0^1 X(-\nu') \exp(-2b/\nu') \nu' \\ \times \left(\frac{\nu + k_0 \tan[(b + z_0)/k_0]}{\nu^2 + k_0^2} - \frac{1}{\nu + \nu'} \right) B(\nu') d\nu' \quad (10)$$

where

$$\bar{f}(c, \nu) = (c - 1)g(c, \nu)X(-\nu)(\nu^2 + k_0^2).$$

Equations (8) and (10) are particular cases of integral equations in which the integral operator depends on a parameter b , and we rewrite them in operator notation as

(i) $c < 1$:

$$\lambda B_{\pm}(\nu) = F(c, \nu, b) + \int_0^1 K_{\pm}(\nu, \nu', b)B_{\pm}(\nu') d\nu' \tag{11}$$

where

$$K_{\pm}(\nu, \nu', b) = \mp \frac{c}{2} f(c, \nu)X(-\nu') \exp(-2b/\nu')\nu' \left(\frac{1}{\nu + \nu'} + h_{\pm}(c, \nu, b) \right) \tag{12}$$

and $F(c, \nu, b)$ is the free term of equation (8).

(ii) $c > 1$:

$$\lambda B(\nu) = \int_0^1 K(\nu, \nu', b)B(\nu') d\nu' \tag{13}$$

with

$$K(\nu, \nu', b) = \frac{c}{2} \bar{f}(c, \nu)X(-\nu') \exp(-2b/\nu')\nu' \left(\frac{\nu + k_0 \tan[(b + z_0)/k_0]}{\nu^2 + k_0^2} - \frac{1}{\nu + \nu'} \right). \tag{14}$$

In the following sections we study these operator equations using the methods of functional analysis.

3. The eigenvalue problem and its inverse

The integral operators generated by the kernels in equations (11) and (13) define a family of operator-valued functions $\mathbb{K}(b)$ depending on the parameter b ; hence the eigenvalue will also be b -dependent (Kato 1966). The variation of λ with b depends on the nature of the eigenvalue spectrum of the operator $\mathbb{K}(b)$ for any fixed b . If $\mathbb{K}(b)$ is an operator in a finite dimensional space, then its spectrum consists only of a finite number of isolated eigenvalues and in such a case the variation of the eigenvalues with a variation of the perturbation parameter is well established (Kato 1966). When the operator acts on an infinite dimensional space, like the present instance of the Banach space of continuous positive functions[†], then the spectrum can consist of a residual and continuous part, beside an (isolated) point spectrum which need not be finite. In such a case, additional restrictions on the operator, such as compactness, eliminate the residual and continuous spectra. If in addition we can prove that there is only a finite system of (isolated) point spectra of the type we wish to investigate (real eigenvalues, for example) then the theory of operators in a finite dimensional space can be applied without further modifications. It is also to be noted that various complications can arise when an infinite number of eigenvalues of $\mathbb{K}(b)$ are considered simultaneously (Kato 1966).

An eigenvalue problem in perturbation theory therefore consists in determining how the eigenvalues and eigenvectors change with the operator, and the case of greatest

[†] The justification of this restriction on $B(\nu)$ is given in § 6.

interest is when the latter depends on the parameter analytically. The inverse of this problem in which we are interested, specifies the eigenvalue (and its variation), and seeks to determine the operators (i.e. b) which produce this eigenvalue. Such problems occur naturally in neutron transport theory and though they have been the subject of study before, mainly in connection with time eigenvalues (Wing 1962, Larsen and Zweifel 1974, Ukai 1965, Vidav 1968), the critical eigenvalue has not been similarly analysed. In the present paper we adopt this approach to the study of the critical problem by a systematic application of the theory of perturbations of linear operators.

It will be seen that operator perturbation theory is a natural setting for the study of such problems which we call the inverse eigenvalue problem. Though problems of this nature have been studied before (Wing 1962, Larsen and Zweifel 1974, Ukai 1965, Vidav 1968), consistent use of perturbation theory does not seem to have been attempted. Our objective will be to show that there is only one real eigenvalue function for the family of integral operators $\mathbb{K}(b)$. This allows us to apply perturbation theory of linear operators in a finite dimensional space.

An elegant and complete mathematical theory exists for the study of such equations if the operator \mathbb{K} is positive, that is provided it leaves a cone \mathbb{C} invariant. A set \mathbb{C} in a linear space is called a cone when it has the following properties: (i) if x and $y \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{R}_+$, the set of positive real numbers, then $\alpha x + \beta y \in \mathbb{C}$ and (ii) if $x \in \mathbb{C}$ and $-x \in \mathbb{C}$ then $x = 0$. The set of non-negative functions in various functions spaces and the set of vectors with non-negative components are examples of cones. Thus the criterion for positivity of an operator can be expressed by saying that \mathbb{K} is positive if for any $x \in \mathbb{C}$, $\mathbb{K}x \in \mathbb{C}$. An integral operator is positive on the cone of non-negative functions if $K(\nu, \nu') > 0$ for all ν and ν' in their domain of definition in \mathbb{R}_+ . An important property of a cone \mathbb{C} in a vector space is the partial order relation defined by it, i.e. if $x \leq y$, then $y - x \in \mathbb{C}$. If \mathbb{K} depends on a parameter b analytically, then different b values define different operators in general. The variation of \mathbb{K} with b is reflected in a variation of λ with b . When $\mathbb{K}(b)$ is holomorphic near $b = b_0$ any finite system of eigenvalues of $\mathbb{K}(b)$ consists of branches of one or several analytic functions which have at most algebraic singularities near $b = b_0$ (Kato 1966). The number of eigenvalues is independent of b , except at a finite number of special values called the exceptional points, at which the number changes. Thus for example (Kato 1966) an operator $\mathbb{K}(b)$ with the matrix representation

$$\mathbb{K}(b) = \begin{pmatrix} 1 & b \\ b & -1 \end{pmatrix}$$

has eigenvalues

$$\lambda_{\pm}(b) = \pm(1 + b^2)^{1/2}$$

which are the two branches of the analytic function $(1 + b^2)^{1/2}$. The exceptional points are those values of b for which $\lambda_+(b) = \lambda_-(b)$ i.e. $b = \pm i$ are the exceptional points. There may be no exceptional points in a certain domain D (for example when the (b) is independent of b) or there may be exceptional points at which different analytic functions—unlike the two branches of the same analytic function as above—are equal. At the exceptional point, the branches of each of the analytic functions constitute a cycle of period equal to the number of branches of the analytic function in question. Of course, there can be more than one such cycle, each corresponding to the branches of different analytic functions at a single exceptional point. In the example considered above, there is one cycle of period 2 at each of the exceptional points $\pm i$.

Further information on the eigenvalues can be had when $\mathbb{K}(b)$ is an integral operator valued function of b . Thus let the positive kernel $K(\nu, \nu', b)$ generate a positive operator \mathbb{K} which leaves a cone of positive functions $B(\nu)$ invariant. Then if $K(\nu, \nu', b)$ be monotonic in b for all $\nu, \nu' \in \mathbb{R}_+$, then the spectral radius $r(b)$ is also monotonic, that is, if $K(\nu, \nu', b_1) > K(\nu, \nu', b_2)$ when $b_1 > b_2$, then $r(b_1) > r(b_2)$. Since \mathbb{K} has been assumed to be positive, it must have, by the Frobenius theorem, a positive real eigenvalue $\lambda(b) = r(b)$ greater in magnitude than the modulus of all other eigenvalues. Thus $\lambda(b)$, like $r(b)$, is a monotonically increasing analytic function of b . In addition it can be shown that there are no other real eigenvalues of \mathbb{K} , and if the problem restricts itself to real eigenvalues only, then we have a well-formulated, especially simple instance of the general case in which the eigenvalue of \mathbb{K} forms only one cycle of period 1 without any exceptional points. Summarising, we can conclude that under the restrictions mentioned above (all of which will be shown to be valid in our case), we have unique monotonically increasing analytic function $\lambda(b)$, as the real eigenvalue of the family of operators $\mathbb{K}(b)$.

The resolvent $R(b, \lambda)$ of the operator $\mathbb{K}(b)$ defined by

$$R(b, \lambda) = (\mathbb{K}(b) - \lambda \mathbb{I})^{-1}$$

is known to be holomorphic in the variables b and λ in each domain in which λ is not an eigenvalue of $\mathbb{K}(b)$. Let $\mathbb{K}(b)$ be a family of compact operators in a Banach space holomorphic for $b \in D$, and suppose λ_0 is an eigenvalue of $\mathbb{K}(b_0)$. Then from the above, we can conclude that there will, in general, be several branches of λ in every region $|\lambda - \lambda_0|$ for sufficiently small $|b - b_0|$. Each of these branches will either cut the $\lambda = \lambda_0$ line for a definite discrete $b = b_i$ or will be identically equal to λ_0 in $|b - b_0|$. Thus either the resolvent $R(b, \lambda_0)$ will exist as a holomorphic function for every $b \neq b_i$ or λ_0 is an eigenvalue for all $b \in D$. Note that for a given λ_0 , the different singular points b_i arise from the different branches of the one or more analytic functions referred to earlier. The above result, variously attributed to Gohberg (Larsen and Zweifel 1974), Smulyan (Vidav 1968) and Atkinson (Kato 1966) is obviously an important one, as it leads to the determination of the unknown parameter b in an eigenvalue problem with a fixed known eigenvalue λ_0 . This problem therefore specifies the eigenvalue and seeks to determine one or more values of b —and hence the specific members of the family $\mathbb{K}(b)$ —for which λ_0 is the eigenvalue. The problem is hence the inverse of the standard eigenvalue problem in which one finds the eigenvalue given the operator, and will be termed the inverse eigenvalue problem. The inverse eigenvalue problem will have a unique solution, that is it will have one and only one value of b , if the eigenvalues form a single cycle of period 1. In the inverse problem posed by equations (8) and (10), $\lambda_0 = 1$, and we require the b for which a non-trivial solution exists. For equation (10) this must be unique if it is to specify the critical thickness, b_c . We will show below that this is indeed the case.

4. The critical slab problem

We will first consider the case of a multiplying medium with no incident neutrons, i.e. equations (13) and (14), and analyse the kernel for (i) positivity in ν and ν' for all b , and (ii) monotonicity in b for all ν and ν' . The operator \mathbb{K} must also be completely

continuous for all $b \in D$. Since ν and ν' are both less than 1, \mathbb{K} will be completely continuous in \mathbb{L}_2 if the kernel has no singularities; this is true in each of the segments

$$0 \leq \frac{b+z_0}{k_0} < \frac{\pi}{2}, \quad n \frac{\pi}{2} < \frac{b+z_0}{k_0} < (n+2) \frac{\pi}{2}, \quad n = 1, 3, \dots \quad (15a, b)$$

Positivity of the kernel will be met if

$$\frac{\nu + k_0 \tan[(b+z_0)/k_0]}{\nu^2 + k_0^2} \geq \frac{1}{\nu + \nu'},$$

that is if

$$\nu' \geq \frac{\nu^2 + k_0^2}{\nu + k_0 \tan[(b+z_0)/k_0]} - \nu.$$

Since $\nu \in (0, 1)$, the above will hold for all ν if

$$\nu' \geq \frac{k_0}{\tan[(b+z_0)/k_0]}.$$

Hence \mathbb{K} will be positive for all b , if $0 \leq \nu < 1$ and ν' is restricted to the subset

$$\nu_L(b) \leq \nu' \leq 1$$

of $(0, 1)$ with

$$\nu_L(b) = \frac{k_0}{\tan[(b+z_0)/k_0]}. \quad (16)$$

For any $c > 1$, equation (16) also fixes a minimum possible value:

$$b \geq k_0 \tan^{-1}(k_0) - z_0 = b_L, \quad (17)$$

say, so that

$$\nu' \leq 1.$$

To show that $K(\nu, \nu', b)$ is monotonic in $b \geq b_L = k_0 \tan^{-1}(k_0) - z_0$, differentiate the kernel with respect to b . Then

$$\frac{\partial K}{\partial b} = -\frac{2}{\nu'} K(\nu, \nu', b) + \alpha(\nu, \nu') \frac{\exp(-2b/\nu')}{\nu^2 + k_0^2} \sec^2\left(\frac{b+z_0}{k_0}\right), \quad (18a)$$

$$\begin{aligned} \frac{\partial^2 K}{\partial b^2} = & 4 \left[\frac{1}{\nu'^2} K(\nu, \nu', b) - \frac{1}{\nu'} \alpha(\nu, \nu') \frac{\exp(-2b/\nu')}{\nu^2 + k_0^2} \sec^2\left(\frac{b+z_0}{k_0}\right) \right. \\ & \left. + \frac{\alpha(\nu, \nu')}{2k_0} \frac{e^{-2b/\nu_0}}{\nu^2 + k_0^2} \sec^2\left(\frac{b+z_0}{k_0}\right) \tan\left(\frac{b+z_0}{k_0}\right) \right]. \quad (18b) \end{aligned}$$

Now because $K(\nu, \nu', b)$ is the kernel of an integral equation, the ranges of ν and ν' must be the same, that is $(\nu_L(b), 1)$. When $b = b_L$, $\nu_L(b) = 1$ and therefore $\nu = \nu' = 1$. Then it is easy to see from (18a) that

$$\left. \frac{\partial K}{\partial b} \right|_{b=b_L} = 0.$$

We now demonstrate that if $\partial K/\partial b = 0$ for any other $b \in (b_L, \frac{1}{2}k_0\pi - z_0)$, which by equation (15a) is the region of interest, then $\partial^2 K/\partial b^2 > 0$. This, together with the fact that $K(\nu, \nu', b) \rightarrow \infty$ as $b \rightarrow \frac{1}{2}k_0\pi - z_0$ proves that $\mathbb{K}(b)$ is monotonic in $b_L \leq b < \frac{1}{2}k_0\pi - z_0$.

If $\partial K/\partial b = 0$ then (18a) gives

$$2\left(\frac{\nu + k_0 \tan[(b + z_0)/k_0]}{\nu^2 + k_0^2} - \frac{1}{\nu + \nu'}\right) = \frac{\nu'}{\nu^2 + k_0^2} \sec^2\left(\frac{b + z_0}{k_0}\right).$$

From (18b) $\partial^2 K/\partial b^2$ is proportional to

$$\frac{\nu + k_0 \tan[(b + z_0)/k_0]}{\nu^2 + k_0^2} - \frac{1}{\nu + \nu'} - \frac{\nu'}{\nu^2 + k_0^2} \sec^2\left(\frac{b + z_0}{k_0}\right) \left(1 - \frac{\nu'}{2k_0} \tan\left(\frac{b + z_0}{k_0}\right)\right).$$

At a possible point of inflection therefore,

$$\frac{\partial^2 K}{\partial b^2} \alpha \frac{\nu'}{\nu^2 + k_0^2} \sec^2\left(\frac{b + z_0}{k_0}\right) \left(-\frac{1}{2} + \frac{\nu'}{2k_0} \tan\left(\frac{b + z_0}{k_0}\right)\right) \alpha \frac{\nu'}{\nu_L} - 1$$

which is positive for $\nu' > \nu_L(b)$. As $\partial k/\partial b|_{b=b_L} = 0$ and $\partial^2 K/\partial b^2|_{b=b_L} \equiv \nu' = \nu_L = 1 = 0$, this implies that there cannot be a turning point of $K(\nu, \nu', b)$ in $b_L < b < \frac{1}{2}k_0\pi - z_0$ other than at which $K(\nu, \nu', b)$ is a minimum. Hence we conclude that for $\nu_L(b) \leq \nu, \nu' \leq 1$, $K(\nu, \nu', b)$ increase monotonically with b , that is the operator $\mathbb{K}_{\text{ap}}(b)$ generated by $K(\nu, \nu', b)$ with $\nu_1(b) \leq \nu, \nu' \leq 1$ increases from zero at $b = b_L$ (when $\nu_L(b) = 1$) to infinity at $b = \frac{1}{2}\pi k_0 - z_0$. Each b therefore defines a unique operator of the family $\mathbb{K}(b)$ and $r(b_1) > r(b_2)$ for $b_1 > b_2$. For the inverse eigenvalue problem to have a unique solution, that is for the existence of only one b_c for $\lambda(b) = \lambda_0 = 1$, it is necessary and sufficient that there exists only one positive eigenvalue function $\lambda(b)$ of $\mathbb{K}(b)$. To show this, we use the Hopf criterion, derivable from the Hopf inequality (Hopf 1963), which states that if the ratio $K(\nu_1, \nu', b)/K(\nu, \nu', b)$ of a positive kernel is bounded and greater than unity for all $\nu, \nu', \nu_1 \in \mathbb{R}_+$, that is if

$$\frac{\max_{\nu_1, \nu'} K(\nu_1, \nu', b)}{\min_{\nu, \nu'} K(\nu, \nu', b)} \leq \theta(b) > 1, \tag{19}$$

then there exists one and only one non-negative solution ϕ_0 of the operator equation $\mathbb{K}(b)\phi = \lambda(b)\phi$ corresponding to the one and only one analytic function $\lambda(b) = \lambda_0(b) > 0$ for all b , i.e $\lambda = \lambda_0, \phi = \phi_0$ is the only positive solution of the eigenvalue equation under consideration. As the ratio of the kernels in (19) is a function of ν and ν' (for a fixed b), we can delete a set of measure zero at which $K(\nu_1, \nu', b)/K(\nu, \nu', b)$ is discontinuous when testing for the criterion (19). This deletion of a set of measure zero will not affect the Riemann integrability of the kernel function and the solution of the integral equation will exist almost everywhere, except at the particular deleted set of measure zero. As an example the integral operator defined by the kernel $K(\nu, \nu') = \nu + \nu', 0 \leq \nu, \nu' \leq 1$ has the eigenvalues -0.077 and 1.077 . The Hopf bound is finite and greater than unity if the points $\nu = \nu' = 0$ are excluded which proves the existence of only one positive eigenvalue greater in modulus than any other eigenvalue. Similarly for the kernel $E_1(\beta|x - x'|)$, x and $x' \in (-b, b)$, encountered in the time dependent one speed equation (Wing 1962), $\theta(\beta)$ is infinite when $x = x'$. This means that there will be more than one positive eigenvalue of the operator in question. Actually, it is known (Wing 1962) that the operator has a finite point spectrum lying on $\beta > 0$.

In the present case the ratio in (19) is

$$\frac{K(\nu_1, \nu', b)}{K(\nu, \nu', b)} = \frac{\bar{f}(c, \nu_1)((\nu_1 + k_0 \tan(b + z_0)/k_0)/(\nu_1^2 + k_0^2) - (\nu_1 + \nu')^{-1})}{\bar{f}(c, \nu)\left(\frac{\nu + k_0 \tan(b + z_0)/k_0}{\nu^2 + k_0^2} - \frac{1}{\nu + \nu'}\right)}$$

and as long as we are in a region where $\mathbb{K}(b)$ is holomorphic, i.e. in the interval given by equation (15), the above will be finite if the point $\nu = 1$ is excluded (because $g(c, 1) = 0$). Thus equation (19) is verified and we conclude that in each strip of analyticity there exists only one analytic single-valued real function $\lambda(b) = r(b)$ and hence $\lambda(b_1) > \lambda(b_2)$ if $b_1 > b_2$. An elementary application of the Gohberg-Atkinson theorem† now tells us that the inverse eigenvalue problem has a unique solution for any chosen $\lambda(b) = \lambda_0$. Our intention is to determine this unique solution in the interval (15a) when $\lambda_0 = 1$, that is to find the critical half-thickness b_c . We note that in order to formulate the problem as above, we had to ensure that the integral operator was positive; this led to equation (16). Thus we do not solve for the exact problem but for the approximate one

$$\mathbb{K}_{ap}(b)\phi(\nu) = \lambda_{ap}(b)\phi(\nu) \tag{20}$$

where

$$\mathbb{K}_{ap}(b)\phi(\nu) = \int_{\nu_L(b)}^1 K(\nu, \nu', b)\phi(\nu') d\nu', \quad \nu_L(b) \leq \nu \leq 1,$$

and $b = b_c$ when $\lambda_{ap}(b) = 1$. The lower limit of integration $\nu_L(b)$ is not very different from zero; it is exactly zero only at the end point critical half thickness. Thus if the method of calculation does not produce too great an error in b_c —this will be found to be true—then the approximate equation (16) will describe the exact one with very little error.

Since $\lambda(b) = r(b)$, we use the following estimate of $r(b)$ (Krasnosel'skii 1972) in order to evaluate $\lambda(b)$:

$$\min_{\nu \in (\nu_L, 1)} \int_{\nu_L(b)}^1 K(\nu, \nu', b) d\nu' \leq \gamma(b) \leq \max_{\nu \in (\nu_L, 1)} \int_{\nu_L(b)}^1 K(\nu, \nu', b) d\nu' \tag{21}$$

and take

$$r(b) = \frac{1}{2}(r_{\min}(b) + r_{\max}(b)) \tag{22}$$

where

$$r_{\min}(b) = \min_{\nu \in (\nu_L, 1)} \int_{\nu_L(b)}^1 K(\nu, \nu', b) d\nu', \tag{22a}$$

$$r_{\max}(b) = \max_{\nu \in (\nu_L, 1)} \int_{\nu_L(b)}^1 K(\nu, \nu', b) d\nu'. \tag{22b}$$

Considering the crudity of the bounds in (21), equations (22) give a surprisingly accurate estimate of the largest eigenvalue of an integral operator, as the following examples show.

(i) $K(\nu, \nu') = \exp(\nu + \nu')$; $\nu, \nu' \in (0, 1)$; $\theta = e > 1$, and only one positive eigenvalue exists whose estimate is given by $\bar{\lambda} = \frac{1}{2}(e^2 - 1) = \lambda$, the exact value.

(ii) $K(\nu, \nu') = \nu = \nu'$; $\nu, \nu' \in (0, 1)$; $\bar{\lambda} = 1.00$, $\lambda = 1.077$.

(iii) $K(\nu, \nu') = \sin(\nu + \nu')$; $\nu, \nu' \in (0, \frac{1}{2}\pi)$; θ is finite when $\nu, \nu' \neq 0$; $\bar{\lambda} = 1.2071$, $\lambda = 1.2854$.

(iv) $K(\nu, \nu') = \frac{1}{2}\nu'(2 - \nu)$; $\nu, \nu' \in (0, 1)$; $\theta = 2$; $\bar{\lambda} = 0.375$, $\lambda = 0.333$. Thus equation (22) can be used with some confidence for obtaining a fairly accurate estimate of the

† We prefer to attribute the theorem to Gohberg-Atkinson for chronological reasons.

largest eigenvalue, or spectral radius, of an integral operator. The integral

$$\begin{aligned}
 I(\nu, b) = & \int_{\nu_L(b)}^1 K(\nu, \nu', b) d\nu' = \frac{c}{2} \bar{f}(c, \nu) \frac{\nu + k_0 \tan((b + z_0)/k_0)}{\nu^2 + k_0^2} \\
 & \times \int_{\nu_L(b)}^1 \nu' X(-\nu') \exp(-2b/\nu') d\nu' \\
 & - \frac{c}{2} \bar{f}(c, \nu) \int_{\nu_L(b)}^1 \frac{\nu'}{\nu + \nu'} X(-\nu') \exp(-2b/\nu') d\nu', \tag{23}
 \end{aligned}$$

necessary in the estimation of $\lambda(b)$, was approximated as

$$\begin{aligned}
 I(\nu, b) = & \frac{c}{2} \exp(-2b) X(-1) \bar{f}(c, \nu) \left[\frac{\nu + k_0 \tan[(b + z_0)/k_0]}{2(\nu^2 + k_0^2)} \right. \\
 & \left. \times (1 - \nu_L^2(b)) - \left(1 - \nu_L(b) - \nu \ln \frac{\nu + 1}{\nu + \nu_L(b)} \right) \right], \quad \nu \in (\nu_L(b), 1). \tag{24}
 \end{aligned}$$

Using (24), the spectral radius was calculated as follows. A value of b was chosen ($b > b_L$) and equation (24) evaluated at twenty-one points in the interval $(\nu_L(b), 1)$. The maximum and minimum of these twenty-one values were then obtained and the spectral radius was calculated according to equation (22). If this was equal to 1 within a specified error limit, ($|r(b) - 1| \leq 10^{-5}$) then this choice of b is the solution to the inverse eigenvalue problem, b_c . If it was not, a new value of b was taken and the process repeated. This convergence in the spectral radius with respect to b was achieved by the method of bisection, which could be adopted because the spectral radius increases monotonically with b . The constants needed in the calculation, $k_0(c)$ and $z_0(c)$, were taken from Case *et al* (1953), the $g(c, \nu)$ function was calculated from its definition, equation (7), and $X(-\nu)$ was obtained at the 21 points by a 21-point Simpson's rule numerical quadrature in

$$X(-\nu) = \exp \left[-\frac{c}{2} \int_0^1 g(c, \nu') \left(1 + \frac{c\nu'}{1 - \nu'^2} \right) \ln(\nu + \nu') d\nu' \right]. \tag{25}$$

The values of the critical half thickness so obtained are shown in table 1. For comparison, the table also lists the P_1 and P_3 results and also the ν_L values at b_c .

The critical thicknesses are significantly improved if instead of using equation (24), the integrations are performed numerically in equation (23). In this case, as in the above, the errors increase with c , and the maximum for $c = 2.0$ is 8.20%. These results

Table 1. Critical half-thickness b_c in units of mean free path

c	Exact	Present (% error)	P_1 (% error)	P_3 (% error)	$\nu(b_c)$
1.02	5.6655	5.6655 (0.000)	5.7519 (-1.5253)	5.6711 (-0.0994)	0.0000
1.05	3.3002	3.3002 (0.000)	3.4034 (-3.438)	3.3065 (-0.193)	0.0000
1.1	2.1134	2.1125 (0.043)	2.2287 (-5.454)	2.1213 (-0.376)	0.0008
1.2	1.2893	1.2855 (0.295)	1.4125 (-9.555)	1.3020 (-0.985)	0.0043
1.4	0.7366	0.7231 (1.833)	0.8581 (-15.54)	0.7577 (-2.860)	0.0137
1.6	0.5120	0.4927 (3.769)	0.6269 (-22.44)	0.3394 (-5.151)	0.0225
1.8	0.3887	0.3634 (6.509)	0.4965 (-27.73)	0.4182 (-7.592)	0.0298
2.0	0.3108	0.2805 (9.749)	0.4121 (-32.59)	0.3420 (-10.05)	0.0359

are not reproduced here, however. The computation time is, of course, much greater than if equation (24) is used.

4.1. Discussion of numerical results

Considering the simplicity of the method adopted here and the complexity of the Fredholm equation (only formal Neumann series solution in their lowest orders of approximation have been considered before; note that we do not need such an explicit solution of the integral equation), the results given here can be considered as encouraging. The maximum error in b_c for practical values of c (< 1.6) is less than 4%, which for a numerical integration of (23) reduces to 2.45%. The kernel has a singularity at the end-point half thickness $b_{ep} = \frac{1}{2}\pi k_0 - z_0$ and as the region of analyticity and positivity of $\mathbb{K}(b)$ is $b_L \leq b \leq b_{ep}$, b_c must be less than b_{ep} . This fact is not reflected in the P_N calculations for which, as the table shows, $b_c > b_{ep}$.

5. The slab albedo problem

We now consider the slab albedo problem, i.e. equations (11) and (12). Very little remains to be 'analysed'. As the kernel $K_+(\nu, \nu', b)$ of equation (12) is always negative for all $\nu, \nu' \in (0, 1)$ and all b , the homogeneous equation for $B_+(\nu)$ (from equation (11)) cannot have a solution in the cone of positive functions. As $B_+(\nu)$ is a linear combination of $A(\nu)$ and $A(-\nu)$ (equation (6)), this implies that the $A(\nu)$ do not exist and by equations (3) and (5) we conclude the non-existence of a_{0+} and a_{0-} . The homogeneous slab albedo problem, therefore, does not have a solution, which means that the problem with an incident neutron source has a unique solution for all b and $c < 1$. It is to be noted that a restriction on the values of b and c as given by McCormick and Mendelson (1964) does not arise here, and the solution exists for all b and $c < 1$. The restriction is necessary if the Neumann series is to converge, not for a solution to exist. We therefore conclude the existence of a unique solution for the slab albedo problem for all slab thicknesses and incident flux of neutrons.

6. Conclusions

The exact critical problem for a slab has been formulated as an (approximate) inverse eigenvalue problem. Though the inverse eigenvalue problem has been studied previously in neutron transport theory in a different context (Larsen and Zweifel 1974, Ukai 1965, Vidav 1968) it does not appear to have been systematised on the basis of perturbation theory of operators. It has been shown in this paper that perturbation theory leads in a natural way to the study of inverse eigenvalue problems. Monotonicity of the eigenvalue parameter is sufficient to guarantee the existence of a discrete solution to the inverse problem. Previous authors (Larsen and Zweifel 1974, Vidav 1968) have deduced the existence of a discrete solution by specifically demonstrating that the eigenvalue function tends either to zero or an infinite limit at one point in the domain of definition of the inverse parameter. This implies that the eigenvalue function cannot be a constant, which in turn leads by the Gohberg-Atkinson theorem to the existence of a discrete solution. Unlike the time eigenvalue problem, there cannot be more than one critical eigenvalue in the region given by equation (15a). Uniqueness of the eigenvalue

function was established through the Hopf criterion. The entire formulation depends on the integral operator being positive with positive eigenvalues. This implies that the functions $B(\nu)$ are also positive. While it is impossible to show this explicitly, the theory given above produces results that are acceptably accurate. Therefore the implications inherent in the theory must be valid, which means that the eigenfunctions $B(\nu)$ of the critical problem must actually be positive. It must also be remembered that we have not analysed the exact problem, but only an approximate one. However, the approximate problem differs insignificantly from the real one and it can be concluded that the hypotheses for the ideal problem hold for the real one also. Numerical values of the critical thickness, obtained here by functional analytic methods in contrast to the classical explicit solution of the operator equation (Mitsis 1963, McCormick and Mendelson 1964), give significantly accurate results in a rather simple and straightforward way. Thus, where the determination of the inverse eigenvalue parameter is concerned—and this, as we know, is often of fundamental importance in transport theory—our approach has the virtue of simplicity both in formulation and in application.

References

- Bowden R L, McCrosson F J and Rhodes E A 1968 *J. Math. Phys.* **9** 753
Case K M, de Hoffmann F and Placzek G 1953 *Introduction to the Theory of Neutron Diffusion* (Los Alamos: Scientific Laboratory)
Case K M and Zweifel P F 1967 *Linear Transport Theory* (Reading, MA: Addison-Wesley)
Hopf E 1963 *J. Math. Mech.* **12** 683
Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)
Krasnosel'skii M A 1972 *Approximate solution of Operator Equations* (Gröningen: Wolters-Noordhoff)
Larsen E W and Zweifel P F 1974 *J. Math. Phys.* **15** 1987
Leonard A and Mulliken T W 1964 *J. Math. Phys.* **5** 399
McCormick N J and Mendelson M R 1964 *Nucl. Sci. Engng.* **20** 462
Mitsis G J 1963 *Nucl. Sci. Engng.* **17** 55
Ukai S 1965 *J. Nucl. Energy* **19** 833
Vidav I 1968 *J. Math. Analysis Applic.* **22** 144
Wing G M 1962 *An Introduction to Neutron Transport Theory* (New York: John Wiley)